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Soliton Propagation in a Nonlinear Resonant Medium

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Abstract. Soliton propagation in a nonlinear resonant medium is considered by using a coupled system of the nonlinear Schrödinger equation and the Maxwell-Bloch equations as a model equation. As in the case of other soliton equations, solutions of the coupled system can be constructed as a quotient of Wronskian-type determinants. From the explicit from of the soliton solutions, the effect of the resonance on the NLS-soliton can be evaluated.

1. Introduction

Recently Maimistov and Manykin proposed a coupled system of the nonlinear Schrödinger (NLS) equation and the Maxwell-Bloch (MB) equations to treat an ultra short pulse propagation in a resonant medium with Kerr nonlinearity[1];

$$\left\{\begin{array}{rcl} E_z &=& i\tilde{c}_1 E_{tt} + i\tilde{c}_2|E|^2E + \tilde{c}_3\langle p\rangle, \\ p_t &=& i\tilde{c}_4\alpha p + \tilde{c}_5 E\eta, \\ \eta_t &=& \tilde{c}_6(Ep^* + E^*p), \end{array}\right. \tag{1}$$

where $\tilde{c}_1, \cdots, \tilde{c}_6$ are real constants, subscripts denote partial derivatives, and $\langle p\rangle$ stands for averaging with respect to inhomogeneous broadening of the resonant frequency;

$$\langle p(z,t;\alpha)\rangle = \int_{-\infty}^{\infty} p(z,t;\alpha)g(\alpha)d\alpha, \qquad \int_{-\infty}^{\infty} g(\alpha)d\alpha = 1.$$

Nakazawa, Yamada and Kubota used eqs.(1) as model equations for pulse propagation in erbium-doped optical fibers and obtained a solution of one-soliton type[2][3].

Imposing the condition,

$$\tilde{c}_1\tilde{c}_5\tilde{c}_6 + \tilde{c}_2 = 0,$$

on eqs.(1) and using a suitable rescaling, we have

$$\left\{\begin{array}{rcl} E_z &=& i\tilde{a}_1\left(\frac{1}{2}E_{tt} + |E|^2E\right) + 2\tilde{a}_2\langle p\rangle, \\ p_t &=& 2i\alpha p + 2E\eta, \\ \eta_t &=& -(Ep^* + E^*p). \end{array}\right. \tag{2}$$

Equations (2) are obtained from the following Lax-type equation,

$$\frac{\partial L}{\partial z} - \frac{\partial A}{\partial t} = [A, L],$$

where L and A are matrix-valued functions of formal indeterminate λ given by

$$
\begin{aligned}
L &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda + \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}, \\
A &= i\tilde{a}_1\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}\lambda \right. \\
&\left. +\frac{1}{2}\begin{pmatrix} |E|^2 & E_t \\ E_t^* & -|E|^2 \end{pmatrix}\right\} + \tilde{a}_2\begin{pmatrix} \left\langle\frac{\eta}{\lambda-i\alpha}\right\rangle & \left\langle\frac{-p}{\lambda-i\alpha}\right\rangle \\ \left\langle\frac{-p^*}{\lambda-i\alpha}\right\rangle & \left\langle\frac{-\eta}{\lambda-i\alpha}\right\rangle \end{pmatrix}.
\end{aligned}
$$

We note that this equation can be regarded as a member of a hierarchy of completely integrable equations[4].

2. Soliton Solutions

By using the notion of the Grassmann formulation of the soliton equations, we know that the solution of eqs.(2) can be constructed as a quotient of determinants[4]. For example, the 1-soliton solution is written as follows;

$$
\begin{aligned}
E(z,t) &= 2\begin{vmatrix} 1 & \mu+i\nu \\ 1 & -\mu+i\nu \end{vmatrix} \Big/ \begin{vmatrix} 1 & e^{-\varphi-i\psi+i\theta} \\ 1 & -e^{\varphi-i\psi+i\theta} \end{vmatrix} \\
&= 2\mu\,\mathrm{sech}\,(\varphi(z,t))\exp\left(i\psi(z,t)-i\theta\right), && \text{(3a)} \\
p(z,t;\alpha) &= \frac{2\mu\left\{\mu\sinh\left(\varphi(z,t)\right)+i(\nu-\alpha)\cosh\left(\varphi(z,t)\right)\right\}\exp\left(i\psi(z,t)-i\theta\right)}{\mu^2\sinh^2\left(\varphi(z,t)\right)+(\nu-\alpha)^2\cosh^2\left(\varphi(z,t)\right)+\mu^2/4}, && \text{(3b)} \\
\eta(z,t;\alpha) &= \frac{\mu^2\sinh^2\left(\varphi(z,t)\right)+(\nu-\alpha)^2\cosh^2\left(\varphi(z,t)\right)-\mu^2/4}{\mu^2\sinh^2\left(\varphi(z,t)\right)+(\nu-\alpha)^2\cosh^2\left(\varphi(z,t)\right)+\mu^2/4}, && \text{(3c)}
\end{aligned}
$$

where $\varphi(z,t)$ and $\psi(z,t)$ are given by

$$
\begin{aligned}
\varphi(z,t) &= 2\mu t + \left\{-4\tilde{a}_1\mu\nu + \int_{-\infty}^{\infty}\frac{2\tilde{a}_2\mu}{\mu^2+(\nu-\alpha)^2}g(\alpha)d\alpha\right\}z + \varphi^{(0)}, && \text{(4a)} \\
\psi(z,t) &= 2\nu t + \left\{2\tilde{a}_1(\mu^2-\nu^2) - \int_{-\infty}^{\infty}\frac{2\tilde{a}_2(\nu-\alpha)}{\mu^2+(\nu-\alpha)^2}g(\alpha)d\alpha\right\}z + \psi^{(0)}. && \text{(4b)}
\end{aligned}
$$

Note that $\varphi^{(0)}$ and $\psi^{(0)}$ are arbitrary parameters, independent of both z and t. From eq.(4a), we see that the velocity of the soliton is given by

$$
V = \left(2\tilde{a}_1\nu - \int_{-\infty}^{\infty}\frac{\tilde{a}_2}{\mu^2+(\nu-\alpha)^2}g(\alpha)d\alpha\right)^{-1}.
$$

In the case $\tilde{a}_1 = 1$ and $\tilde{a}_2 = 0$, the 1-soliton solution (3) coincides with that of the NLS equation[5], and in the case $\tilde{a}_1 = 0$ and $\tilde{a}_2 = 1$, it coincides with that of the MB equation[6].

The 2-soliton solution is constructed as a quotient of 4×4-determinants with Wronskian structure. Here we give the explicit form of the E-field only;

$$
E(z,t) = \frac{4\left[e^{i(\psi_1-\theta_1)}(A\cosh\varphi_2 + iB\sinh\varphi_2) - e^{i(\psi_2-\theta_2)}(\bar{A}\cosh\varphi_1 + iB\sinh\varphi_1)\right]}{C\cosh(\varphi_1+\varphi_2) + \bar{C}\cosh(\varphi_1-\varphi_2) - 4\mu_1\mu_2\cos(\psi_1-\psi_2-\theta_1+\theta_2)}, \quad \text{(5)}
$$

where A, $\bar{A}$, B, C and $\bar{C}$ are real constants given by

$$\begin{aligned} A &= \mu_1\left[\mu_1^2-\mu_2^2+(\nu_1-\nu_2)^2\right], \quad & \bar{A} &= \mu_2\left[\mu_1^2-\mu_2^2-(\nu_1-\nu_2)^2\right], \\ B &= 2\mu_1\mu_2(\nu_1-\nu_2), & & \\ C &= (\mu_1-\mu_2)^2+(\nu_1-\nu_2)^2, & \bar{C} &= (\mu_1+\mu_2)^2+(\nu_1-\nu_2)^2. \end{aligned}$$

The phases $\varphi_j(z,t)$ and $\psi_j(z,t)$ $(j=1,2)$ are obtained by substituting μ_j, ν_j into μ, ν of eqs.(4) respectively.

If we take the values of the parameters μ_j and ν_j which reduce the phases $\varphi_1(z,t)$ and $\varphi_2(z,t)$ to

$$\varphi_1(z,t) = const. \times \varphi_2(z,t)$$

for all z and t, then the solution (5) represents a bound state of two solitons. For the NLS case, it coincides with the the so-called "N=2 soliton"[5], which pulsates with some frequency. The period of the pulsation is sometimes called "soliton period". While the pulsation in the NLS case is symmetric (Fig.1), it becomes asymmetric in the general NLS-MB case ($\tilde{a}_1, \tilde{a}_2 \neq 0$) (Fig.2). This difference between the NLS case and the NLS-MB case comes from their dispersion relations (See eqs.(4)).

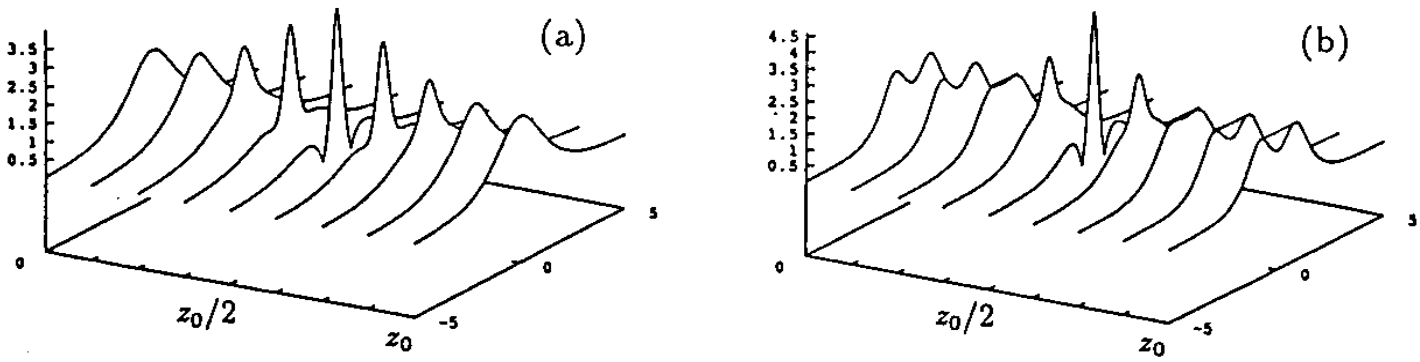

Fig.1. Evolution of the breather-type bound state solution with $\tilde{a}_1 = 1$, $\tilde{a}_2 = 0$ (NLS case) and $g(\alpha) = \delta(\alpha)$; (a) $\mu_1 = 3/2$, $\mu_2 = 1/2$, $\nu_1 = \nu_2 = 0$. (b) $\mu_1 = 3/2$, $\mu_2 = 4/5$, $\nu_1 = \nu_2 = 0$. (z_0 denotes the soliton period.)

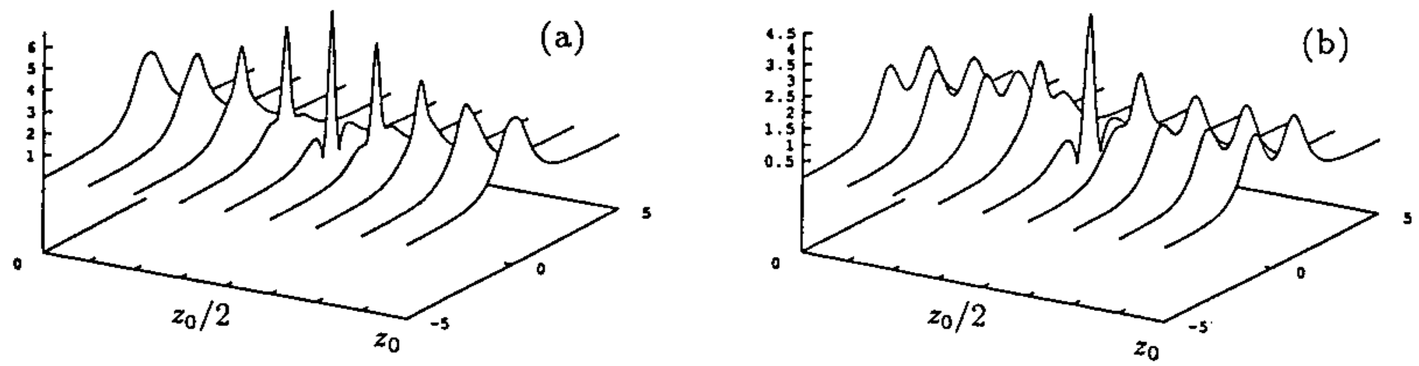

Fig.2. Evolution of the breather-type bound state solution with $\tilde{a}_1 = 1$, $\tilde{a}_2 = 1$ (NLS-MB case) and $g(\alpha) = \delta(\alpha)$; (a) $\mu_1 = \sqrt{3}/2$, $\mu_2 = 11\sqrt{129}/50$, $\nu_1 = 1/2$, $\nu_2 = 2/25$. (b) $\mu_1 = \sqrt{3}/2$, $\mu_2 = \sqrt{31}/4$, $\nu_1 = 1/2$, $\nu_2 = 1/4$. (z_0 denotes the soliton period.)

3. Conclusion

In conclusion, we have obtained the explicit form of the 1- and 2-soliton solutions for the NLS-MB system and showed their behavior. We have found that the bound state of two solitons shows the asymmetric pulsation in the NLS-MB case. So far as the authors know, this type of pulsation has not been reported yet. We hope that this phenomenon would be observed in a real optical system.

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